

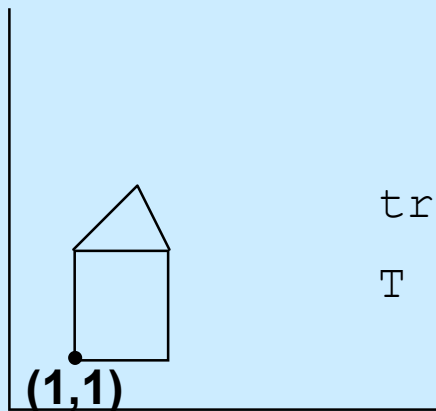
Computer Graphics: Transformations

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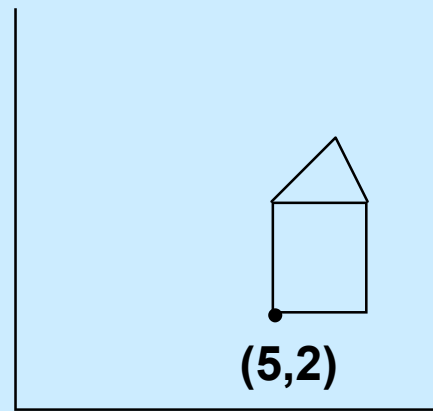
Two-Dimensional Transformations

- Transformations are used to change the size, position, or orientation of objects lying in a coordinate system. They are primarily used for modeling and viewing.
- The transformations that prove most useful are *translation*, *scaling* and *rotation*.
- Translation moves an object from one position to another. To translate a point P by vector T , we can use

$$P' = P + T = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} x + dx \\ y + dy \end{pmatrix}$$



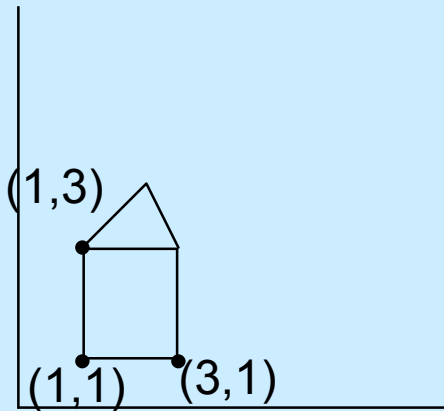
translate by
 $T = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$



Two-Dimensional Transformations

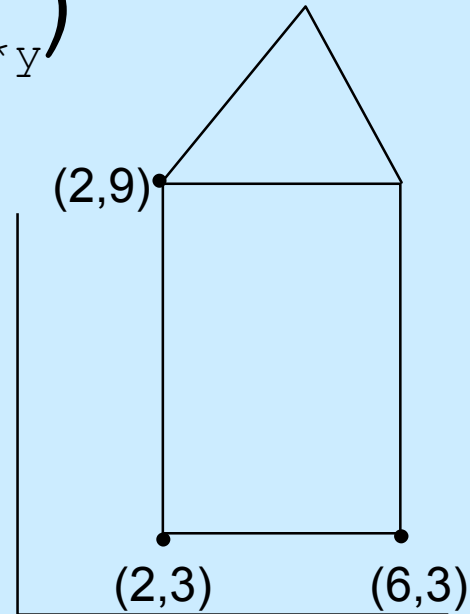
- Scaling changes the size of an object. If its proportion is maintained ($s_x = s_y$), it is called *uniform scaling*. Otherwise it is called *differential scaling*. The matrix S below scales around the origin $(0,0)$.

$$P' = SP = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_x * x \\ s_y * y \end{pmatrix}$$



scale by

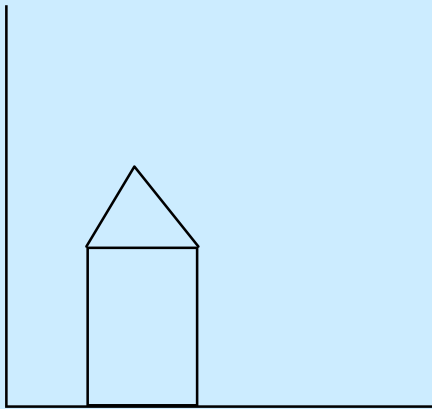
$$s = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$



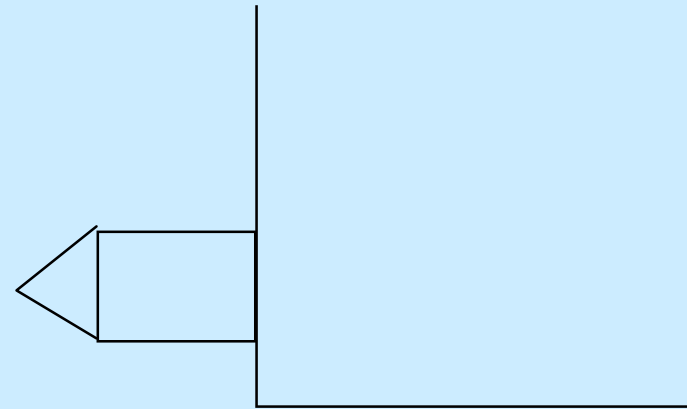
Two-Dimensional Transformations

- **Rotation turns an object around a fixed point. To rotate point P by b degrees counter-clockwise around the origin:**

$$P' = RP = \begin{pmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos(b) - y \sin(b) \\ x \sin(b) + y \cos(b) \end{pmatrix}$$



rotate by
90 degrees



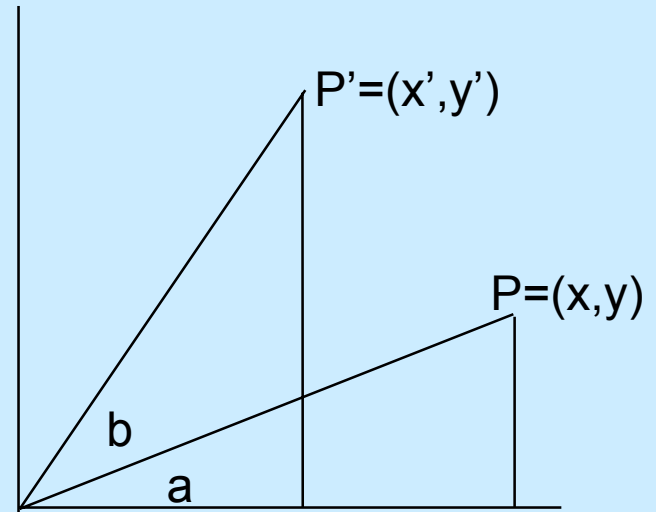
Two-Dimensional Transformations

- The rotation matrix follows from trigonometry:

$$x = r \cos(a); \quad y = r \sin(a)$$

$$\begin{aligned} x' &= r \cos(a + b) \\ &= r (\cos(a) \cos(b) - \sin(a) \sin(b)) \\ &= r \cos(a) \cos(b) - r \sin(a) \sin(b) \\ &= x \cos(b) - y \sin(b) \end{aligned}$$

$$\begin{aligned} y' &= r \sin(a + b) \\ &= r \cos(a) \sin(b) + r \sin(a) \cos(b) \\ &= x \sin(b) + y \cos(b) \end{aligned}$$



Rotation of point P by
b degrees.

Homogeneous Coordinates and 2D Transformations

- The problem with the previous transformations is that translation is achieved by vector addition, whereas scaling and rotation are achieved by matrix multiplication. We would like all transformations to be accomplished using only matrix multiplication, so that transformations can be *composed*. Thus, instead of having to perform a sequence of n transformations one at a time, we could compose these n transformations into a single matrix and apply that matrix once.
- To work with homogeneous coordinates, each point (x, y) in the plane is augmented by a third component w , yielding (x, y, w) . The relationship between the two representations is simple:
 - The homogeneous point (x, y, w) corresponds to the Cartesian point $(x/w, y/w)$, where w is non-zero.
 - In particular, the homogeneous point $(x, y, 1)$ corresponds to the point (x, y) in the plane.
- Using homogeneous coordinates, every 2-dimensional transformation in the plane is represented by a 3x3 matrix.

Homogeneous Coordinates and 2D Transformations

- Here are the 3x3 matrices for 2D translation, scale, and rotation:

**Translation by
(dx, dy):**

$$T_{(dx, dy)} = \begin{pmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{pmatrix}$$

**Scaling by
(sx, sy) about
the origin:**

$$S_{(sx, sy)} = \begin{pmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

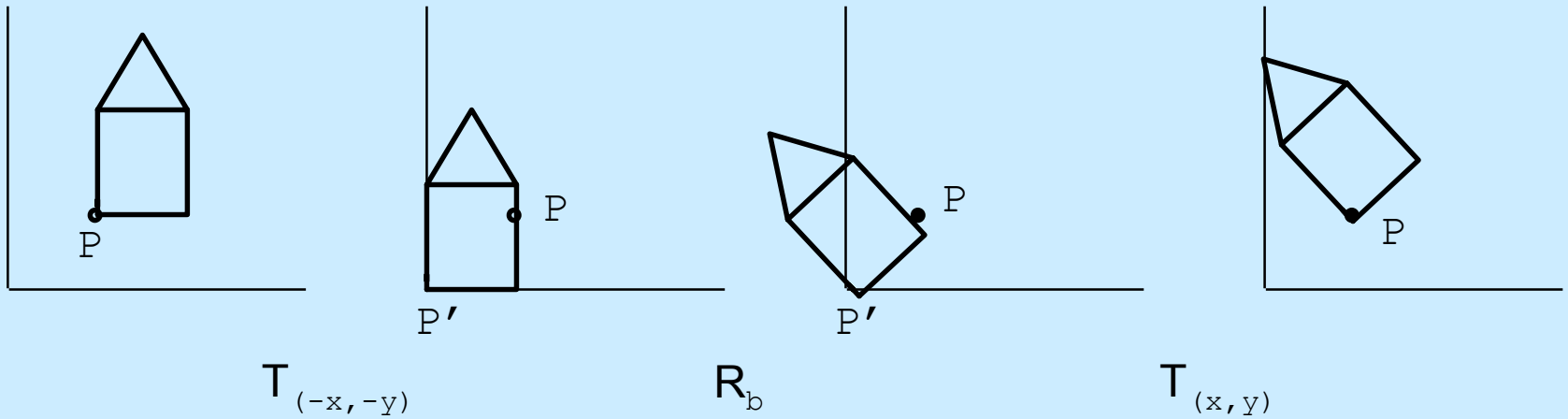
**Rotation by
b degrees around
the origin:**

$$R_b = \begin{pmatrix} \cos(b) & -\sin(b) & 0 \\ \sin(b) & \cos(b) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Composition of Transformations

- **Associativity of matrix multiplication permits us to compose transformation matrices:** $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
- **Here is an example. To rotate a point Q about an arbitrary point P , use**

$$Q' = T_{(x,y)} \cdot R_b \cdot T_{(-x,-y)} \cdot Q$$



$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(b) & -\sin(b) & 0 \\ \sin(b) & \cos(b) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(b) & -\sin(b) & -x\cos(b)+y\sin(b)+x \\ \sin(b) & \cos(b) & -x\sin(b)-y\cos(b)+y \\ 0 & 0 & 1 \end{pmatrix}$$

Composition of Transformations

- To achieve this rotation, we can apply the resulting composite matrix:

$$\begin{pmatrix} \cos(b) & -\sin(b) & -x\cos(b)+y\sin(b)+x \\ \sin(b) & \cos(b) & -x\sin(b)-y\cos(b)+y \\ 0 & 0 & 1 \end{pmatrix}$$

- Applying this single matrix is more efficient than applying the three matrices from which it is composed in succession. This efficiency is critical when we consider that it is often necessary to transform many thousands of points by the same transformation.

Homogeneous Coordinates and 3D Transformations

- We can also use homogeneous coordinates to represent transformations in 3-dimensional space. Everything is lifted up by one dimension: A transformation in space is represented by a 4x4 matrix, and a point in space is represented by a 4-component tuple whose last coordinate is 1:
 - The homogeneous point $(x, y, z, 1)$ corresponds to the point (x, y, z) in space.
- The remaining slides show the matrices corresponding to translation, scaling, and rotation in space. Because rotation can be around any one of the three major axes (the x , y , or z axis), it is necessary to define three different rotation matrices, one per axis.

Homogeneous Coordinates and 3D Transformations

- The 4x4 matrices for 3D translation and scaling are as follows:

**Translation by
(dx, dy, dz):**

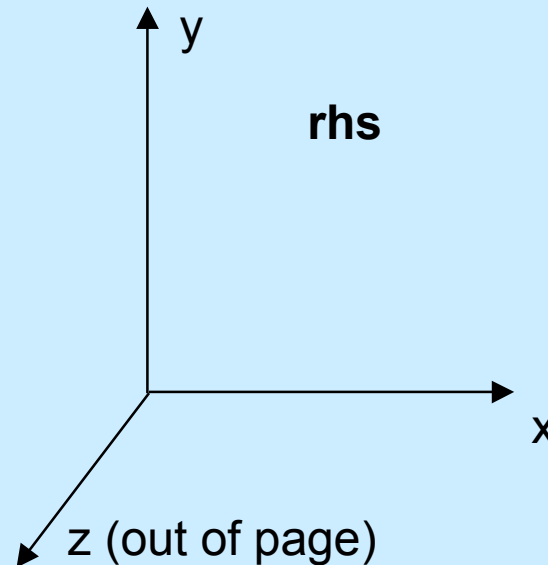
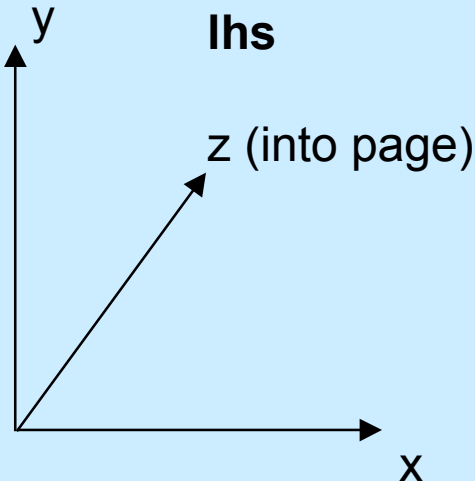
$$T_{(dx, dy, dz)} = \begin{pmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**Scaling by
(sx, sy, sz):**

$$S_{(sx, sy, sz)} = \begin{pmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Homogeneous Coordinates and 3D Transformations

- In three dimensions, three different rotations are possible: one around each axis.
- To describe sense of rotation consistently, it is necessary to select the handedness of the coordinate system.
- Under a right-handed system (rhs), a *counter-clockwise* rotation of 90 degrees around the z axis rotates the x axis into the y axis, around the x axis rotates the y axis into the z axis, and around the y axis rotates the z axis into the x axis.
- Under a left-handed system (lhs), a *clockwise* rotation of 90 degrees accomplishes the same.



3D Rotations in Homogeneous Coordinates in a Right-Handed Coordinate System

Rotation by b
CCW around the
 z -axis:

$$R_{b, z} = \begin{pmatrix} \cos(b) & -\sin(b) & 0 & 0 \\ \sin(b) & \cos(b) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation by b
CCW around the
 x -axis:

$$R_{b, x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(b) & -\sin(b) & 0 \\ 0 & \sin(b) & \cos(b) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation by b
CCW around the
 y -axis:

$$R_{b, y} = \begin{pmatrix} \cos(b) & 0 & \sin(b) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(b) & 0 & \cos(b) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$